Grade 6 Order of Operations

6.N.9	
Explain and apply the order of operations, excluding exponents (limited to whole numbers).	1. Demonstrate and explain with examples why there is a need to have a standardized order of operations. 2. Apply the order of operations to solve multi-step problems with or without technology (e.g., computer, calculator).

Clarification of the outcome:

- ✦ The outcome concerns understanding how to process an arithmetic expression involving whole numbers, with or without brackets.
- ✦ Note that a set of brackets is not an arithmetic operation. It is a container.
- ✦ This is the only occurrence of order of operations in the Manitoba K-8 curriculum. The topic is revisited in grade 9 and includes expressions having exponents. It is an important topic for understanding algebraic manipulation. That topic also begins in grade 9.
- ✦ You likely learned BEDMAS (or PEDMAS) in relation to order of operations. However that approach to teaching order of operations does not transfer well to understanding algebraic manipulation and can easily lead to making errors when working with arithmetic expressions.

Here are two examples.

The correct answer for example #1 is about 3.6.

The correct answer for example #2 is 27/7

or 3 6/7.

$$\sqrt{4+9} = 2 + 3 = 5$$ Example #1

$$\frac{15+12}{5+2} = \frac{\overset{3}{\cancel{15}} + \overset{6}{\cancel{12}}}{\cancel{5} + \cancel{2}} = 3 + 6 = 9$$ Example #2

Required close-to-at-hand prior knowledge:

- ✤ Whole number arithmetic skills

SET SCENE stage

Presenting an arithmetic expression has at least two ways to process is a reasonable approach to setting the scene.

The problem task to present to students:

Present students with the following situation:

> Joe and Harry were arguing about a skill-testing question that was part of a Safeway shopping contest. Winning the prize involved having the correct number on the prize stub and answering the skill-testing question. The question was: 5 + 2 x 6. Joe's stub had the winning number. He just needed to figure out the answer to the question.
>
> Joe thought the answer to 4 + 2 x 3 was 18. Harry disagreed. He thought the answer was 10.

After presenting the situation, ask students to decide which answer is correct and why.

Comments:

BEDMAS is the equivalent of "math for dummies". The following lesson does not develop BEDMAS (**B**rackets-**E**xponents-**D**ivision-**M**ultiplication-**A**ddition-**S**ubtraction) for three reasons. [Refer to Clarification of outcome' for the first 2 reasons.] The third reason is that students too easily think BEDMAS is a mathematical law. IT IS NOT. You do not have to use BEDMAS to get the correct answer. Consider, for example, the arithmetic expression, '**$2 \times 3 + 4 \times (5 + 1 + 2 \times 6) + 5 + 2^3 + 7$**'. This expression can be worked out as shown below, a method that involves reversing the BEDMAS order. If you do not believe the answer below is correct, use BEDMAS to obtain an answer and compare the two answers.

$2 \times 3 + 4 \times (5 + 1 + 2 \times 6) + 12 + 2^3$ [Add 5 and 7 to get 12.]
$6 + 4 \times (5 + 1 + 2 \times 6) + 12 + 2^3$ [Do 2 x 3 to get 6.]
$18 + 4 \times (5 + 1 + 2 \times 6) + 2^3$ [Add 6 and 12 to get 18.]
$18 + 4 \times 5 + 4 \times 1 + 4 \times 2 \times 6 + 2^3$ [Multiply each term inside the bracket by 4]
$18 + 20 + 4 + 48 + 2^3$ [Do the multiplications indicated in the above line.]
$42 + 8 \times 6 + 2^3$ [Add 18, 20, 4, and 48 to get 90.]
$90 + 8$ [2 x 2 x 2 is 8.]
98 [Add 90 & 8]

Do not get the impression that "anything goes" when processing expressions. The above method pays attention to the hierarchy of operators, a matter that underlies BEDMAS. The upcoming lesson develops the hierarchy of operators, a way of thinking that facilitates understanding order of operations.

DEVELOP stage

Activity 1: Revisits SET SCENE and addresses achievement indicator 1.

Discuss students' opinions about the answer to 4 + 2 x 3. Ask the critical questions: "*Do we add first or multiply first. Which order gives an answer that makes sense?*"

Tell students they are about to learn how to figure out answers to more complex arithmetic expressions but first let's see if Joe or Harry is thinking correctly.

Tell students the following story:

> *Rocky the Squirrel was busy this fall storing acorns for the winter because the Rodent Almanac had predicted a cold long winter. Before breakfast, Rocky stored 4 acorns at the base of an old oak tree. He then scurried over to a hollow in a pine tree where he stored 2 piles of acorns. Each pile had 3 acorns in it. How many acorns had Rocky stored before breakfast?*

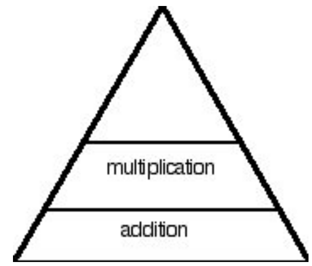
Ask students to draw a picture of what Rocky did. Ensure they draw something like what is shown here.

- ✦ Ask students to count the acorns in the picture. Ensure they count 10. Ask them who is correct. Joe or Harry? Ensure they realize Harry is correct and that the arithmetic expression '4 + 2 x 3' is represented by the picture showing 10 acorns.
- ✦ Discuss why there is a need to have an order to doing arithmetic and why the order cannot be 'do whatever you want'.

Activity 2: Revisits SET SCENE and addresses achievement indicator 2.

- Revisit Joe and Harry's conundrum by **w**riting the expression 4 + 2 x 3 on the board and circling the '2'. Point out that '2' has a choice. It can belong with the '4' and thus be added to 4 OR it can belong with the '3' and thus be multiplied by 3. Remind them that the correct answer is 10 and that the only way to get it is if the '2' belongs with the '3'.

- Ask students, WHEN THERE IS A CHOICE between addition and multiplication, which operation is BOSS? Draw and discuss the first stage of the BOSS triangle (see diagram). Ensure students realize that multiplication is boss when there is a choice between multiplication and addition.



- Underline '2 x 3' in '4 + 2 x 3'. Discuss that '2 x 3' is a multiplication bundle and that a useful thing to do when figuring out an answer to an arithmetic expression is to find and underline the multiplication bundles.

- Present three slightly more complex arithmetic expressions involving addition and multiplication similar to the example here (4 + 5 x 3 + 2 + 7 x 1). Ask students to underline the multiplication bundles and then figure out the answer to each expression. Ensure students can recognize multiplication bundles and they continue to realize that multiplication is BOSS over addition ONLY WHEN there is a CHOICE between the two operations.

- Present three arithmetic expressions involving addition and multiplication in which there are two consecutive additions (refer to the underlined additions in the example: e.g. <u>4 + 5</u> + 2 x 3 + 6 + 3 x 4). Ask students to underline the multiplication bundles and then figure out the answer to each expression. Ensure students can recognize multiplication bundles and they continue to realize that multiplication is BOSS over addition ONLY WHEN there is a CHOICE between the two operations. In relation to the example here, ensure students realize that the <u>4 + 5</u> part can be added BEFORE any of the multiplications are done because the '4' and '5' are not part of a multiplication bundle.

Activity 3: Addresses achievement indicator 2 and practice.

- ✦ Ask students to underline the multiplication bundles and then to figure out the answer to 3 + 4 x 5 + 2 + 2 x 6 in two different ways. Discuss the correct answer (37) and different ways to obtain it (e.g. add 3 and 2 first, add 20, add 11; e.g. do 4 x 5 and 2 x 6 first, then add 3 and add 2).

- ✦ Ask students to underline the multiplication bundles and then to figure out the answer to 8 + 5 + 3 x 2 + 7 + 2 x 6 + 1 + 5 x 6 + 2 x 2 in two different ways. Discuss the ways.

> $8 + 5 + \underline{3 \times 2} + 7 + \underline{2 \times 6} + 1 + \underline{5 \times 6} + \underline{2 \times 2} = 73$
>
> There are many ways to obtain the answer. Here are two of the ways:
>
> Way #1
>
> Add 8, 5, 7, and 1, getting 21.
>
> Do the multiplication for each multiplication bundle, getting 6, 12, 30, and 4
>
> Add 21 to 6, 12, 30, and 4, getting 73.
>
> Way #2
>
> Proceed from right to left in turn: 4 + 30 + 1 + 12 + 7 + 6 + 5 + 8, getting 73.

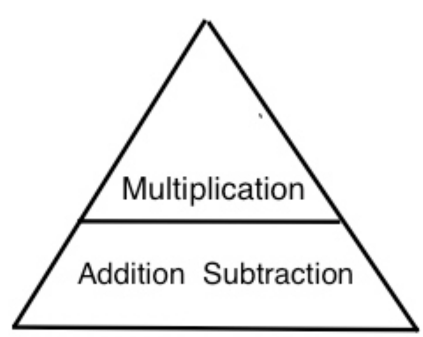
- ✦ Repeat for a couple of similar questions (have students underline the multiplicative bundles each time). Ask for and discuss their work.

Activity 4: Addresses achievement indicator 2 & practice (via non-routine problem solving).

- ✦ Ask students to obtain the numbers from 1 to 4, using only the numbers 0 and 1 (as many times as desired), and the arithmetic operations of addition and multiplication (as many times as desired). Brackets are <u>not</u> allowed. [e.g. 1 = 0 x 1 + 1; 2 = 1 x 1 + 1 x 1; 3 = 1 x 1 + 1 + 1; 4 = 1 x 1 + 1 x 1 + 1 + 1]

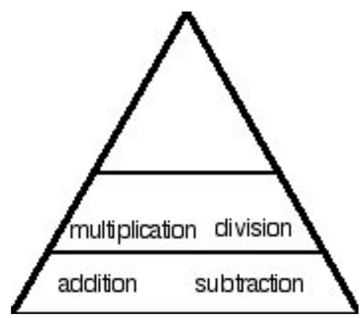
Activity 5: Addresses achievement indicator 2.

Ask students where subtraction might go in the BOSS triangle. Have them make up expressions that involve only addition and subtraction to test their theories. Make up stories for selected expressions, as needed, to help students realize that addition and subtraction are equal in bossiness. Place subtraction in the BOSS triangle (see diagram).



Activity 6: Addresses achievement indicators 1, 2, 4, 5, and 6 & practice.

- Provide students with an arithmetic expression containing one addition and one division (e.g. $12 + 8 \div 2$). Ask students to obtain an answer. Discuss their solutions. Make up a story about the expression (e.g. *Rocky found 12 acorns in a pile of leaves. He took these acorns home. He also found 8 acorns in a pail. He split these acorns into two equal groups and took one of the groups home. How many acorns did Rocky take home?*) to help them realize that 16, not 10, makes sense as the answer. Ask if there are any 'glued' bundles in the expression (expect uncertainty). Discuss the relationship between multiplication and division to help them realize that division also "glues" numbers together into a bundle.
- Circle the number that has a choice (the '8', in this example, could belong to '12' or '2'). Ask students if addition or division is BOSS when there is a choice between the two operations. Ensure that students understand that 'division' is boss over 'addition' when there is a choice. Discuss again '$8 \div 2$' as a bundle that is "glued" together. The bundle must be "unglued" by doing the division before it can be added to something.
- Ensure that students understand that division must also be boss over subtraction because addition and subtraction are equal in bossiness.
- Ask students where division belongs in the BOSS triangle. [Is multiply boss over division, or division boss over multiply, or are they equal in bossiness?] Ask students to make up and obtain answers to arithmetic expressions that only involve multiplication and division to test their theories about which operation is boss. [Ensure students make up expressions where multiplication appears first (e.g. $5 \times 4 \div 2$) and where division appears first (e.g. $24 / 4 \times 2$) so that students do not get the false impression that one operation is boss over the other.] Lead students to realize that division and multiplication are equal in bossiness and that when involved in a chain, the best way to deal with the matter is to do the arithmetic in left to right order. Place division in the triangle.



Note:

In the expressions that students make up, a situation such as: '$24 \div 4 \times 2$' will occur. If this expression is done in the order of writing, you obtain 12 as the answer ($24 \div 4 = 6$, $6 \times 2 = 12$). If the multiplication is done first, you obtain 3 as the answer ($4 \times 2 = 8$, $24 \div 8 = 3$). Which answer is correct? [12 is correct.]

The question of multiplication or division being boss is tricky. Consider again $24 \div 4 \times 2$. If you translate the division into a multiplication by using a fraction form (not a grade 6 outcome), then there is no issue (division and multiplication are clearly equal bosses). $4 \div 4 \times 2$ becomes $24 \times 1/4 \times 2$, and this expression can be done any which way (e.g. $24 \times 2 = 48$, $48 \times 1/4 = 12$; e.g. $24 \times 1/4 = 6$, $6 \times 2 = 12$).

The form '$24 \div 4 \times 2$' says that 24 is divided by 4, not by 4×2. For grade 6 students, he easiest way around this matter is to develop the rule that when division and multiplication are involved in a chain, then do the arithmetic in left to right order (thus for $24 \div 4 \times 2$: $24 \div 4 = 6$, $6 \times 2 = 12$).

Activity 7: Addresses achievement indicator 2 and practice.

Tell students that they are going to determine answers for more difficult skill-testing questions for winning a prize. Provide a variety of arithmetic expressions that involve at least three operations. Do not include brackets. Here are three examples:

- 2 x 3 + 5 + 7 - 3 + 14 ÷ 2 + 3 x 2 (answer is 28)
- 10 + 20 + 30 ÷ 6 x 2 + 4 - 1 + 5 x 3 (answer is 58)
- 12 ÷ 2 x 3 ÷ 9 - 1 + 5 + 3 x 4 (answer is 18)

Ensure students underline multiplication/division bundles before doing the arithmetic. Ask for and discuss solutions.

Activity 8: Addresses achievement indicator 2.

Present two expressions involving one addition and one multiplication where the only difference is that one expression contains brackets while the other does not (e.g. 2 x 3 + 4 and 2 x (3 + 4)). Ask students to make up a story for each expression and obtain an answer for each expression based on its story. Discuss the role of the brackets.

Present other pairs of expressions involving one or more additions/subtractions and one or more multiplications/divisions where the only difference is that one expression contains brackets while the other does not. Ask students to obtain an answer to the expressions. Ask for and discuss results. Ensure that students realize that brackets can change the meaning of an expression but they do not have to be done first.

To help negate the misconception about brackets as 'do me first' creatures, obtain the answer to an expression containing brackets by using the distributive principle (see below).

$$2 \times (3 + 4) = 2 \times 3 + 2 \times 4 = 6 + 8 = 14$$

Compare this answer to that obtained by "doing what is inside the brackets first" (see below).

$$2 \times (3 + 4) = 2 \times 7 = 14$$

Also help negate the misconception about brackets being 'do me first' creatures by obtaining answers to expressions that have several additions outside the brackets (see below).

$$4 + 5 + 3 \times (1 + 6) = 9 + 3 \times (1 + 6) = 9 + 3 \times 7 = 9 + 21 = 30$$

Ask if 'brackets' belong in the BOSS triangle. Ensure students understand that brackets cannot belong because brackets are not arithmetic operations. All brackets can do is to sometimes change the meaning of an expression. Provide examples of where meaning is not changed and examples of where meaning is changed (see below). Discuss the situations.

No change: 2 x 3 + 4 x 5 compared to (2 x 3) + (4 x 5)

Change: 2 + 3 x 4 compared to (2 + 3) x 4

Activity 9: Addresses achievement indicator 2 & practice (via non-routine problem solving).

Ask students to solve the following problem:

> Using four 4s each time, and at least one of addition/subtraction/multiplication/division/brackets, make an arithmetic expression that results in each number from 0 to 3.

Discuss students' solutions. Below are some possible solutions.

- $0 = 4 - 4 + 4 - 4$
- $1 = 4 - 4 + 4 \div 4$
- $2 = 4 \div 4 + 4 \div 4$
- $3 = (4 + 4 + 4) \div 4$).

Activity 10: Addresses achievement indicator 2 & practice.

Organize students into groups of two. Student #1 in each pair makes up two order of operations expressions that could be skill-testing questions for winning a prize. Student #2 obtains answers to each expression. Student #1 confirms whether the answers are correct. Then the two students switch roles.

Activity 11: Assessment of teaching.

- Provide students with the arithmetic expression: '5 + 3 x 4'. Ask them to determine the answer and to explain by using a story and a diagram for the story why their answer is correct.
- Provide students with two arithmetic expressions - one not too complex, the other somewhat complex (see below for examples). Ask them to determine the answers.
  - $3 \times 4 + 2 - 3 + 15 \div 3 + 2$
  - $10 + 3 + 4 \times (5 + 2 - 1) - 7 + 20 \div 5 \times 2 + 2 \times (3 - 1)$

If all is well with the assessment of teaching, engage students in PRACTICE (the conclusion to the lesson plan).

One example of partially well-designed worksheet follows.

The worksheet contains a sampling of question types. More questions of each type are needed for a well-designed worksheet.

The MAINTAIN stage follows the sample worksheets.

Question 1.

Make a story and draw a picture for it that shows why $2 + 3 \times 4 = 14$.

Question 2.

Determine the answer to each expression.

a) $7 + 3 - 2$

b) $15 + (7 - 3) - 1$

c) $15 + 7 - (3 - 1)$

Question 3.

Determine the answer to each expression.

a) $15 \div 3 \times 2$

b) $6 \times 4 \div 2 \div 3$

Question 4.

Determine the answer to each expression.

a) $4 \times 6 + 4 \times 2 - 3 + 5 \times 7$

b) $(12 - 4) \div 4$

c) $(5 + 1) \times 2$

d) $3 + 7 - 1 + 2 \times (5 + 3) - 2 \times 4$

Question 5.

Determine if the following equality statements are true.

a) $(4 + 3) \times 2 = 5 \times 3 - 1$

b) $3 \times 4 \div 2 = 10 - 3 \times 2$

c) $5 + 12 = 5 + 1 + 12 - 1$

d) $3 + 2 \times 5 + 4 \times (1 + 2 \times 3) = 2 \times 15 \div 3 + 3 \times 7 + 20 \div 2$

MAINTAIN stage

Mini-task example

Every so often:

- Present a not too complex arithmetic expression [e.g. 2 x 4 + 5 x (7 - 2 x 2)] and ask students to determine its value.

Rich-task example

Ask students to solve the following problem:

"*Using four 7s each time, and at least one of addition/subtraction/multiplication/division/ brackets, try to make an expression for the whole numbers whose answer is from 0 to 60. Here is an example for the answer 5: 5 = 7 - (7 + 7) ÷ 7*

Note:

It is not possible to obtain an expression for all the numbers from 0 to 60. Below are some of the possible ones.

- ☑ One solution for 3 is: *(7 + 7 + 7) ÷ 7*
- ☑ One solution for 13 is: 7 + 7 - 7 ÷ 7
- ☑ One solution for 56 is: (7 + 7 ÷ 7) x 7
- ☑ One solution for 1 is: 7 ÷ 7 x 7 ÷ 7
- ☑ One solution for 49 is: 7 + 7 x 7 - 7
- ☑ One solution for 14 is: 7 ÷ 7 x (7 + 7)

Comments

This is a rich-task because it involves a complex task concerning non-routine problem solving.